A mean value formula for elliptic curves *

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Abstract

It is proved in this paper that for any point on an elliptic curve, the mean value of x-coordinates of its n-division points is the same as its x-coordinate and that of y-coordinates of its n-division points is n times of its y-coordinate.

Keywords: elliptic curves, point multiplication, division polynomial

1 Introduction

Let K be a field with $\mathrm{char}(K) \neq 2, 3$ and let $\overline{K}$ be the algebraic closure of K. Every elliptic curve E over K can be written as a classical Weierstrass equation

$$E : y^2 = x^3 + ax + b$$

with coefficients a, $b \in K$. A point Q on E is said to be smooth (or non-singular) if $\left(\frac{\partial f}{\partial x}|_Q, \frac{\partial f}{\partial y}|_Q\right) \neq (0, 0)$, where $f(x, y) = y^2 - x^3 - ax - b$. The point

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multiplication is the operation of computing

$$nP = \underbrace{P + P + \cdots + P}_{n}$$

for any point $P \in E$ and a positive integer n. The multiplication-by-n map

$$\begin{array}{rccc} [n]: & E & \to & E \\ & P & \mapsto & nP \end{array}$$

is an isogeny of degree n^2. For a point $Q \in E$, any element of $[n]^{-1}(Q)$ is called an n-division point of Q. Assume that $(\mathrm{char}(K), n) = 1$. In this paper, the following result on the mean value of the x, y-coordinates of all the n-division points of any smooth point on an elliptic curve is proved.

Theorem 1. *Let E be an elliptic curve defined over K, and let $Q = (x_Q, y_Q) \in E$ be a point with $Q \neq \mathcal{O}$. Set*

$$\Lambda = \{P = (x_P, y_P) \in E(\overline{K}) \mid nP = Q\}.$$

Then

$$\frac{1}{n^2} \sum_{P \in \Lambda} x_P = x_Q$$

and

$$\frac{1}{n^2} \sum_{P \in \Lambda} y_P = n y_Q.$$

According to Theorem 1, let $P_i = (x_i, y_i), i = 1, 2, \cdots, n^2$, be all the points such that $nP = Q$ and let λ_i be the slope of the line through P_i and Q, then $y_Q = \lambda_i(x_Q - x_i) + y_i$. Therefore,

$$n^2 y_Q = \sum_{i=1}^{n^2} \lambda_i \cdot (\sum_{i=1}^{n^2} x_i)/n^2 - \sum_{i=1}^{n^2} \lambda_i x_i + \sum_{i=1}^{n^2} y_i.$$

Thus we have

$$y_Q = \frac{\sum\limits_{i=1}^{n^2} \lambda_i}{n^2} \cdot \frac{\sum\limits_{i=1}^{n^2} x_i}{n^2} - \frac{\sum\limits_{i=1}^{n^2} \lambda_i x_i}{n^2} + \frac{\sum\limits_{i=1}^{n^2} y_i}{n^2} = \overline{\lambda_i} \cdot \overline{x_i} - \overline{\lambda_i x_i} + \overline{y_i},$$

where $\overline{\lambda_i},\ \overline{x_i},\ \overline{\lambda_i x_i},\ \overline{y_i}$ are the average values of the variables $\lambda_i, x_i, \lambda_i x_i$ and y_i, respectively. Therefore,

$$Q = (x_Q, y_Q) = (\overline{x_i},\ \overline{\lambda_i} \cdot \overline{x_i} - \overline{\lambda_i x_i} + \overline{y_i}) = \left(\overline{x_i},\ \frac{1}{n}\overline{y_i}\right).$$

Remark: The discrete logarithm problem in elliptic curve E is to find n by given P, $Q \in E$ with $Q = nP$. The above theorem gives some information on the integer n.

2 Proof of Theorem 1

To prove Theorem 1, define division polynomials [4] $\psi_n \in \mathbb{Z}[x, y, a, b]$ on an elliptic curve $E : y^2 = x^3 + ax + b$, inductively as follows:

$$\begin{aligned}
\psi_0 &= 0, \\
\psi_1 &= 1, \\
\psi_2 &= 2y, \\
\psi_3 &= 3x^4 + 6ax^2 + 12bx - a^2, \\
\psi_4 &= 4y(x^6 + 5ax^4 + 20bx^3 - 5a^2x^2 - 4abx - 8b^2 - a^3), \\
\psi_{2n+1} &= \psi_{n+2}\psi_n^3 - \psi_{n-1}\psi_{n+1}^3, \text{ for } n \geq 2, \\
2y\psi_{2n} &= \psi_n(\psi_{n+2}\psi_{n-1}^2 - \psi_{n-2}\psi_{n+1}^2), \text{ for } n \geq 3.
\end{aligned}$$

It can be checked easily by induction that the ψ_{2n}'s are polynomials. Moreover, $\psi_n \in \mathbb{Z}[x, y^2, a, b]$ when n is odd, and $(2y)^{-1}\psi_n \in \mathbb{Z}[x, y^2, a, b]$ when n is even. Define the polynomial

$$\phi_n = x\psi_n^2 - \psi_{n-1}\psi_{n+1}$$

for $n \geq 1$. Then $\phi_n \in \mathbb{Z}[x, y^2, a, b]$. Since $y^2 = x^3 + ax + b$, replacing y^2 by $x^3 + ax + b$, one have that $\phi_n \in \mathbb{Z}[x, a, b]$. So we can denote it by $\phi_n(x)$. Note that, $\psi_n\psi_m \in \mathbb{Z}[x, a, b]$ if n and m have the same parity. Furthermore, the division polynomials ψ_n have the following properties.

Lemma 2.

$$\psi_n = nx^{\frac{n^2-1}{2}} + \frac{n(n^2-1)(n^2+6)}{60}ax^{\frac{n^2-5}{2}} + \textit{ lower degree terms},$$

when n is odd, and

$$\psi_n = ny\left(x^{\frac{n^2-4}{2}} + \frac{(n^2-1)(n^2+6)-30}{60}ax^{\frac{n^2-8}{2}} + \textit{ lower degree terms}\right),$$

when n is even.

Proof. We prove the result by induction on n. It is true for $n < 5$. Assume that it holds for all ψ_m with $m < n$. We give the proof only for the case for odd $n \geq 5$. The case for even n can be proved similarly. Now let $n = 2k+1$ be odd, where $k \geq 2$. If k is even, then by induction,

$$\begin{aligned}
\psi_k &= ky(x^{\frac{k^2-4}{2}} + \tfrac{(k^2-1)(k^2+6)-30}{60}ax^{\frac{k^2-8}{2}} + \cdots), \\
\psi_{k+2} &= (k+2)y(x^{\frac{k^2+4k}{2}} + \tfrac{(k^2+4k+3)(k^2+4k+10)-30}{60}ax^{\frac{k^2+4k-4}{2}} + \cdots), \\
\psi_{k-1} &= (k-1)x^{\frac{k^2-2k}{2}} + \tfrac{(k-1)(k^2-2k)(k^2-2k+7)}{60}ax^{\frac{k^2-2k-4}{2}} + \cdots, \\
\psi_{k+1} &= (k+1)x^{\frac{k^2+2k}{2}} + \tfrac{(k+1)(k^2+2k)(k^2+2k+7)}{60}ax^{\frac{k^2+2k-4}{2}} + \cdots,
\end{aligned}$$

By substituting y^4 by $(x^3+ax+b)^2$, we have

$$\psi_{k+2}\psi_k^3 = k^3(k+2)\left(x^{2k^2+2k} + \frac{4(k+1)(k^3+k^2+10k+3)}{60}ax^{2k^2+2k-2} + \cdots\right),$$

and

$$\psi_{k-1}\psi_{k+1}^3 = (k-1)(k+1)^3x^{2k^2+2k} + \frac{4k(k-1)(k^3+2k^2+11k+7)(k+1)^3}{60}ax^{2k^2+2k-2} + \cdots.$$

Therefore

$$\begin{aligned}
\psi_{2k+1} &= \psi_{k+2}\psi_k^3 - \psi_{k-1}\psi_{k+1}^3 \\
&= (2k+1)x^{2k^2+2k} + \tfrac{(2k+1)(4k^2+4k)(4k^2+4k+7)}{60}ax^{2k^2+2k-2} + \cdots \\
&= (2k+1)x^{\frac{(2k+1)^2-1}{2}} + \tfrac{(2k+1)((2k+1)^2-1)((2k+1)^2+6)}{60}ax^{\frac{(2k+1)^2-5}{2}} + \cdots
\end{aligned}$$

The case when k is odd can be proved similarly. □

The following corollary follows immediately from Lemma 2.

Corollary 3.

$$\psi_n^2 = n^2 x^{n^2-1} - \frac{n^2(n^2-1)(n^2+6)}{30} a x^{n^2-3} + \cdots ,$$

and

$$\phi_n = x^{n^2} - \frac{n^2(n^2-1)}{6} a x^{n^2-2} + \cdots .$$

□

Proof of Theorem 1: Define ω_n as

$$4y\omega_n = \psi_{n+2}\psi_{n-1}^2 - \psi_{n-2}\psi_{n+1}^2.$$

Then for any $P = (x_P, y_P) \in E$, we have ([4])

$$nP = \left(\frac{\phi_n(x_P)}{\psi_n^2(x_P)}, \frac{\omega_n(x_P, y_P)}{\psi_n(x_P, y_P)^3} \right).$$

If $nP = Q$, then $\phi_n(x_P) - x_Q\psi_n^2(x_P) = 0$. Therefore, for any $P \in \Lambda$, the x-coordinate of P satisfies the equation $\phi_n(x) - x_Q\psi_n^2(x) = 0$. From Corollary 3, we have that

$$\phi_n(x) - x_Q\psi_n^2(x) = x^{n^2} - n^2 x_Q x^{n^2-1} + \text{lower degree terms}.$$

Since $\sharp\Lambda = n^2$, every root of $\phi_n(x) - x_Q\psi_n^2(x)$ is the x-coordinate of some $P \in \Lambda$. Therefore

$$\sum_{P\in\Lambda} x_P = n^2 x_Q$$

by Vitae's Theorem.

Now we prove the mean value formula for y-coordinates. Let K be the complex number field $\mathbb{C}$ first and let ω_1 and ω_2 be complex numbers which are linearly independent over $\mathbb{R}$. Define the lattice

$$L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z}\},$$

and the Weierstrass $\wp$-function by

$$\wp(z) = \wp(z, L) = \frac{1}{z} + \sum_{\omega\in L, \omega\neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

For integers $k \geq 3$, define the Eisenstein series G_k by

$$G_k = G_k(L) = \sum_{\omega \in L, \omega \neq 0} \omega^{-k}.$$

Set $g_2 = 60G_4$ and $g_3 = 140G_6$, then

$$\wp^{'}(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

Let E be the elliptic curve given by $y^2 = 4x^3 - g_2x - g_3$. Then the map

$$\begin{array}{rcl} \mathbb{C}/L & \rightarrow & E(\mathbb{C}) \\ z & \mapsto & \left(\wp(z), \wp^{'}(z)\right), \\ 0 & \mapsto & \infty, \end{array}$$

is an isomorphism of groups $\mathbb{C}/L$ and $E(\mathbb{C})$. Conversely, it is well known [4] that for any elliptic curve E over $\mathbb{C}$ defined by $y^2 = x^3 + ax + b$, there is a lattice L such that $g_2(L) = -4a, g_3(L) = -4b$ and there is an isomorphism between groups $\mathbb{C}/L$ and $E(\mathbb{C})$ given by $z \mapsto \left(\wp(z), \frac{1}{2}\wp^{'}(z)\right)$ and $0 \mapsto \infty$. Therefore, for any point $(x, y) \in E(\mathbb{C})$, we have $(x, y) = \left(\wp(z), \frac{1}{2}\wp^{'}(z)\right)$ and $n(x, y) = \left(\wp(nz), \frac{1}{2}\wp^{'}(nz)\right)$ for some $z \in \mathbb{C}$.

Let $Q = \left(\wp(z_Q), \frac{1}{2}\wp^{'}(z_Q)\right)$ for a $z_Q \in \mathbb{C}$. Then for any $P_i \in \Lambda$, $1 \leq i \leq n^2$, there exist integers j, k with $0 \leq j, k \leq n - 1$, such that

$$P_i = \left(\wp\left(\frac{z_Q}{n} + \frac{j}{n}\omega_1 + \frac{k}{n}\omega_2\right), \frac{1}{2}\wp^{'}\left(\frac{z_Q}{n} + \frac{j}{n}\omega_1 + \frac{k}{n}\omega_2\right)\right).$$

Thus

$$\sum_{j,k=0}^{n-1} \wp\left(\frac{z_Q}{n} + \frac{j}{n}\omega_1 + \frac{k}{n}\omega_2\right) = n^2\wp(z_Q)$$

which comes from $\sum\limits_{i=1}^{n^2} x_i = n^2 x_Q$. Differential for z_Q, we have

$$\sum_{j,k=0}^{n-1} \wp^{'}\left(\frac{z_Q}{n} + \frac{j}{n}\omega_1 + \frac{k}{n}\omega_2\right) = n^3\wp'(z_Q).$$

That is

$$\sum_{i=1}^{n^2} y_i = n^3 y_Q.$$

Secondly, let K be a field of characteristic 0 and let E be the elliptic curve over K given by the equation $y^2 = x^3 + ax + b$. Then all of the equations describing the group law are defined over $\mathbb{Q}(a, b)$. Since $\mathbb{C}$ is algebraically closed and has infinite transcendence degree over $\mathbb{Q}$, $\mathbb{Q}(a, b)$ can be considered as a subfield of $\mathbb{C}$. Therefore we can regard E as an elliptic curve defined over $\mathbb{C}$. Thus the result follows.

At last assume that K is a field of characteristic p. Then the elliptic curve can be viewed as one defined over some finite field $\mathbb{F}_q$, where $q = p^m$ for some integer m. Without loss of generality, let $K = \mathbb{F}_q$ for convenience. Let $K' = \mathbb{Q}_q$ be an unramified extension of the p-adic numbers $\mathbb{Q}_p$ of degree m, and let $\overline{E}$ be an elliptic curve over K' which is a lift of E. Since $(n, p) = 1$, the natural reduction map $\overline{E}[n] \to E[n]$ is an isomorphism. Now for any point $Q \in E$ with $Q \neq \mathcal{O}$, we have a point $\overline{Q} \in \overline{E}$ such that the reduction point is Q. For any point $P_i \in E(\overline{K})$ with $nP_i = Q$, its lifted point $\overline{P}_i$ satisfies $n\overline{P}_i = \overline{Q}$ and $\overline{P}_i \neq \overline{P}_j$ whenever $P_i \neq P_j$. Thus

$$\sum_{i=1}^{n^2} y(\overline{P}_i) = n^3 y(\overline{Q})$$

since K' is a field of characteristic 0. Therefore the formula $\sum_{i=1}^{n^2} y_i = n^3 y_Q$ holds by the reduction from $\overline{E}$ to E. □

Remark:

(1) The result for x-coordinate of Theorem 1 holds also for the elliptic curve defined by the general Weierstrass equation $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.

(2) The mean value formula for x-coordinates was given in the first version of this paper [1] with a slightly complicated proof. The formula for y-coordinates was conjectured by D. Moody based on [1] and numerical examples in a personal email communication [2].

(3) Recently, a mean value formula for twisted Edwards curves was given by D. Moody [3].

3 An application

Let E be an elliptic curve over K given by the Weierstrass equation $y^2 = x^3 + ax + b$. Then we have a non-zero invariant differential $\omega = \frac{dx}{y}$. Let $\phi \in \mathrm{End}(E)$ be a nonzero endomorphism. Then $\phi^*\omega = \omega \circ \phi = c_\phi \omega$ for some $c_\phi \in \overline{K}(E)$ since the space Ω_E of differential forms on E is a 1-dimensional $\overline{K}(E)$-vector space. Since $c_\phi \neq 0$ and $\mathrm{div}(\omega) = 0$, we have

$$\mathrm{div}(c_\phi) = \mathrm{div}(\phi^*\omega) - \mathrm{div}(\omega) = \phi^*\mathrm{div}(\omega) - \mathrm{div}(\omega) = 0.$$

Hence c_ϕ has neither zeros nor poles and $c_\phi \in \overline{K}$. Let φ and ψ be two nonzero endomorphisms, then

$$c_{\varphi+\psi}\omega = (\varphi + \psi)^*\omega = \varphi^*\omega + \psi^*\omega = c_\varphi\omega + c_\psi\omega = (c_\varphi + c_\psi)\omega.$$

Therefore, $c_{\varphi+\psi} = c_\varphi + c_\psi$. For any nonzero endomorphism ϕ, set $\phi(x, y) = (R_\phi(x), yS_\phi(x))$, where R_ϕ and S_ϕ are rational functions. Then

$$c_\phi = \frac{R'_\phi(x)}{S_\phi(x)},$$

where $R'_\phi(x)$ is the differential of $R_\phi(x)$. Especially, for any positive integer n, the map $[n]$ on E is an endomorphism. Set $[n](x, y) = (R_n(x), yS_n(x))$. From $c_{[1]} = 1$ and $[n] = [1] + [(n-1)]$, we have

$$c_{[n]} = \frac{R'_n(x)}{S_n(x)} = n.$$

For any $Q = (x_Q, y_Q) \in E$, and any

$$P = (x_P, y_P) \in \Lambda = \{P = (x_P, y_P) \in E(\overline{K}) \mid nP = Q\},$$

we have $y_P = \frac{y_Q}{S_n(x_P)}$. Therefore, Theorem 1 gives

$$\sum_{P\in\Lambda} \frac{1}{S_n(x_P)} = \sum_{P\in\Lambda} \frac{y_P}{y_Q} = \frac{1}{y_Q}\sum_{P\in\Lambda} y_P = n^3.$$

Thus

$$\sum_{P\in\Lambda} \frac{1}{R'_n(x_P)} = \sum_{P\in\Lambda} \frac{1}{n \cdot S_n(x_P)} = \frac{1}{n}\sum_{P\in\Lambda} \frac{1}{S_n(x_P)} = n^2,$$

and

$$\sum_{P\in\Lambda} \frac{x_Q}{R'_n(x_P)} = x_Q \sum_{P\in\Lambda} \frac{1}{R'_n(x_P)} = n^2 x_Q = \sum_{P\in\Lambda} x_P.$$

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